THE EXACT FRIEDMAN TEST AND MULTIPLE COMPARISON PROCEDURE IN MACRO FORM

By

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DESCRIPTIVE ABSTRACT

This paper discusses the development of a macro to perform the EXACT Friedman test and an associated multiple comparison procedure. Also provided is an overview of the Friedman test and the multiple comparison used. SAS code for the macro is provided.

INTRODUCTION

Friedman's test is a nonparametric test for a randomized block design. This test analyzes k treatments across b blocks of k experimental units each (see, for example, Daniel, 1978). SAS has, as yet, not included this nonparametric test in a PHOG. David Ipe (1987) illustrated how to use an F approximation to get SAS to perform this test. This paper illustrates an alternative way to compute Friedman's test statistic. This alternative way does not use sophisticated programming techniques but does illustrate how SAS may be used to solve problems. This work was performed while the first author was enrolled in an undergraduate course on use of the statistical software packages (taught by the second author).

FRIEDMAN'S NONPARAMETRIC TEST FOR A RANDOMIZED BLOCK DESIGN

Friedman's test is the nonparametric analog of the usual randomized block analysis of variance (ANOVA) and was developed by Nobel Laureate (in economics) Milton Friedman. If the underlying responses may not be assumed to follow a normal distribution or if only the ranks of the responses are available, Friedman's test may be used to perform an appropriate analysis. It is assumed that b blocks (i.e., sets of k homogeneous experimental units each) are available. Within each block, each of the k treatments is applied to one experimental unit. The resulting data may be tabulated in the following array.

<table>
<thead>
<tr>
<th>BLOCK</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>k</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>X_{11}</td>
<td>X_{12}</td>
<td>...</td>
<td>X_{1k}</td>
</tr>
<tr>
<td>2</td>
<td>X_{21}</td>
<td>X_{22}</td>
<td>...</td>
<td>X_{2k}</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>b</td>
<td>X_{b1}</td>
<td>X_{b2}</td>
<td>...</td>
<td>X_{bk}</td>
</tr>
</tbody>
</table>

Several assumptions are needed for Friedman's test. First, it is assumed that the above layout holds (and no X_{ij}'s are missing). Second, it is assumed that measurements within each block represent continuous variables. Third, these measurements within a block can be ranked. Fourth, there are no interactions between treatments and blocks. Unlike the classical randomized block ANOVA, no assumption about an underlying error structure is necessary.

The null hypothesis for Friedman's test is that the probability distribution is the same for each of the k treatments. The alternative hypothesis is that at least two of the treatments have probability distributions that differ with respect to their location parameters.

In order to conduct the test, the data are first assigned ranks (R_{ij}) within each block. That is, the smallest observation is assigned rank 1, the next smallest rank two, and so on. Ties are addressed below. The ranks are next summed across treatment (denoted R_{j}). Friedman's test is then computed as

$$F = \left\{ \frac{12}{bk \cdot (k+1)} \sum R_{j}^2 \right\} - 3b \cdot (k+1)$$

where the summation is made across the k treatments. If the null hypothesis is true, then F has an approximate chi-square distribution with (k-1) degrees of freedom.

If there are ties in the values within a block, Friedman's test is modified in two ways. First, the midrank (i.e., the average of the ranks) is used for each tied value. Second, F is divided by the following quantity

$$1 - \frac{\sum \tau_{i} t_{i}}{bk \cdot (k^2 - 1)}$$

where the summation is taken over each block and where \tau_{i} is given by

$$\tau_{i} = \sum_{j} \left( t_{ij}^2 \right) - \sum_{j} t_{ij}^2$$

for t_{ij} equal to the number of observations tied for a given rank in the ith block. The approximate chi-square distribution with (k-1) degrees of freedom holds for this adjusted statistic also.
If the null hypothesis of equal distributions for all treatments is rejected, multiple comparisons using the sum of the rank for each of the treatments may be of interest. Comparisons among all possible pairs of means may be made as follows. First the critical value is computed as

\[ z \left( \frac{bk(k+1)}{6} \right)^{1/2} \]

where \( z \) is the standard normal critical value such that the probability a standard normal random variable exceeds \( z \) equals \( (0.1)^{1/2} \) for desired significance level. Next, the absolute value of the difference between the sums of the ranks of each pair of treatments is computed. Any absolute difference that exceeds the above critical value indicates the two treatment distributions are different.

EXAMPLE OF FRIEDMAN'S TEST

As an example, Friedman's test is computed for the data presented in Figure 1. This design has nine blocks and three treatments. The sum of the ranks for the three treatments are 15.5, 9, and 25.5 for treatments A, B, and C respectively. The value of \( F \) is 15.5 and the corresponding rejection region is \( F > cv \) where \( cv \) is the upper 100(1- \( 0.05 \) ) critical value for a chi-square distribution with two degrees of freedom (eg, for \( =0.05 \), \( cv=5.99 \)). The data includes one tied value (block four). Using midranks and computing the correction factor (equal to .9722), \( F \) is divided by .9772 to give a corrected test statistic of 15.94. As this exceeds the above critical value, the null hypothesis is rejected and we conclude there is a difference among the distributions of the three treatments.

Multiple comparisons are computed next. Using a significance level of 0.05 with three treatments, the desired \( z \)-score is 2.394 and the critical value for comparison between sums of ranks is 10.157. Thus treatments A and B and treatments B and C are significantly different from one another (absolute value of the differences in ranks are 10.5 and 16.5, respectively) while treatments A and C are not significantly different from one another.

SAS MACROS FOR FRIEDMAN'S TEST

Two MACROS were written to perform Friedman's test. The first of these generates an array of ranks within each of the blocks. The second MACRO performs the actual analysis including multiple comparisons. These two MACROS could easily be put together into a single MACRO, but are separated for ease of understanding. The two MACROS are listed in APPENDIX I and the resulting output is given in APPENDIX II.

MACRO RANKSUM begins with unranked data in the array as described above and produces the array of ranks. The data are entered using each block as a record and the \( k \) treatment values as variables. The values are to be ranked across treatments within blocks (that is, across variable). PROC TRANSPOSE is used to transpose the data set (so that ranks are computed across records for single variables), PROC BANK is used to generate the ranks of the values, and PROC TRANSPOSE is used again to return the ranks to the original array. The ranked data set is saved as a permanent data set (called SASDATA.RANKTRAN).

MACRO FRIEDMAN performs the actual Friedman's test using the ranks stored in the permanent data set SASDATA.RANKTRAN. First the number of ties are computed within each block. Next the ranks are summed and the uncorrected Friedman's statistic is computed. The correction factor for ties and the corrected statistic is computed next. Exact p-values are computed for both the uncorrected and corrected statistics. Finally, multiple comparisons are performed for significance level set by the MACRO call. All results are printed including whether or not each pair of means is significantly different using the least significant difference test described above.

CONCLUSIONS

The MACROS given here enable one to perform Friedman's test for a nonparametric randomized block analysis of variance. More importantly, these MACRO illustrate how SAS may be used to perform analyses for which PROCS are not provided. Because of the flexibility of SAS, its PROCS such as TRANSFORM and RANK which perform data manipulations that may be used without performing specific analyses, and because of its MACRO applications, SAS provides a programming language-like environment for the solution of statistical problems.

REFERENCES


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APPENDIX I

MACRO 1

%MACRO RANKSUM (DATASET, A B, C);
/* THIS MACRO IS DESIGNED TO RANK DATA THAT HAS BEEN ENTERED IN A COLUMNAR FORMAT. THE MACRO TRANSPOSES DATA BY THE TREATMENTS, RANKS, AND THEN THE RANKS. NOTES THAT THE RANGE OF THE COL VARIABLES IS ACTUALLY THE NUMBER OF BLOCK VARIABLES. */
PROC TRANSPOSE DATA = &DATASET; /* TRANSPOSE DATA */
VAR &A &B &C;
PROC RANK; /* RANK DATA */
VAR COL1 - COL9;
PROC TRANSPOSE OUT = SASDATA.RANKTRAN; /* TRANSPOSE, CREATE SAS DATASET */
VAR COL1 - COL9;
%MEND RANKSUM;

MACRO 2

%MACRO FRIEDMAN (A, B, C, TRT = X, BLK = Y, ALPHA = Z);
/* FRIEDMAN TEST PROCEDURE */
/* THIS MACRO COMPUTES THE TIES ACROSS THE TREATMENTS, COMPUTES A TIES CORRECTION FACTOR, CHI-SQUARE FRIEDMAN IHTH AND WITHOUT THE CORRECTION FACTOR, AND THE EXACT P-VALUES FOR EACH. A MULTIPLE COMPARISON PROCEDURE IS THEN PERFORMED ON THE TREATMENTS. */
DATA CONSTANT; /* DATASET CONTAINING CONSTANS */
K = &TRT;
B = &BLK;
ALPHA = &ALPHA;
DATA TIES; /* COMPUTES TIES ACROSS TREATMENTS FOR EACH BLOCK */
SET SASDATA.RANKTRAN;
SUMT3 = 0;
SUMT = 0;
T = 0;
IF (&A = &B = &C) /* COMPUTE THE NUMBER OF TIES */
THEN T = 3;
ELSE IF ((&A = &B) OR (&A = &C) OR (&B = &C))
SUMT3 = SUMT3 + (T * T * T);
SUMT = SUMT + T;
PROC MEANS SUM NOPRINT; /* SUM THE TIES */
VAR SUMT3 SUMT;
OUTPUT OUT = SUMTIES SUM = SUMT3SUM SUMTSUM;
PROC MEANS SUM NOPRINT; /* SUM ALL RANKS AND OUTPUT SET */
VAR &A &B &C;
OUTPUT OUT = SUMRANKS SUM = &A.SUM &B.SUM &C.SUM;

DATA FRIEDMAN; /* COMPUTE FRIEDMAN TEST STATISTICS */
SET SUMRANKS; /* GET TOTAL SUM OF SQUARES */
TOTSUMSQ = &A.SUM**2 + &B.SUM**2 + &C.SUM**2;

SET SUMTIES; /* GET SUM OF TI */
SUMT = SUMT3SUM - SUMT3SUM;

SET CONSTANT; /* GET CONSTANTS */
CORRTIES = 1 - (SUMT / (B * K * (K**2) -1))); /* CORRECTION FACTOR */
FR = (12 / (B * K * (K + 1))) * TOTSUMSQ - (3 * B • (K + 1));

P_FR = 1 - PROBCHI (X2FR, K-1);
FRADJ = X2FR / CORRTIES;

P_FRADJ = 1 - PROBCHI (X2FRCT, K-1);

PROC PRINT;
TITLE1 '==================================================================================';
TITLE2 'FRIEDMAN TEST';
TITLE3 'NONPARAMETRIC RANDOMIZED BLOCK DESIGN';
TITLE4 '=============================================';
TITLE5 '';
TITLE6 '';
TITLE7 '1***1*'*'**'*****************************************1**1*1**1******';
TITLE8 'I j';
TITLE9 'I';

DATA MULTCOMP; /* PERFORM THE MULTIPLE COMPARISONS */
SET CONSTANT;
ZPROB = ALPHA / (K * (K -1)); /* GET Z */
ZSTAND = PROBIT(ZPROB);
Z = ABS(ZSTAND);

RHS Z * SQRT ((B * K * (K +1)) / 6); /* GET RIGHT HAND SIDE*/

SET SUMRANKS;

&A.&B = ABS(&A.SUM - &B.SUM); /* COMPUTE ABSOLUTE DIFFERENCES */
&A.&C = ABS(&A.SUM - &C.SUM);
&B.&C = ABS(&B.SUM - &C.SUM);
SIG = 'SIGNIFICANCE ';
NOSIG = 'NO SIGNIFICANCE';
IF (AA.AB ¹ RHS) THEN AA.VSAB. = SIG; ELSE AA.VSAB. = NOSIG;

IF (AA.AC ¹ RHS) THEN AA.VSAC. = SIG; ELSE AA.VSAC. = NOSIG;

PROC PHWT;
TITLE1 'DETERMINE SIGNIFICANCE';
TITLE2 '*';
TITLE3 '* NONPARAMETRIC MULTIPLE COMPARISON PROCEDURE';
TITLE4 '*';
TITLE5 '* LEAST SIGNIFICANT DIFFERENCES';
TITLE6 '*';
TITLE7 '*********************************';
TITLE8 ' ';
TITLE9 ' ';

VAR K H ALPHA Z RHS AA.SUM AB.SUM AC.SUM AA.AB AA.AC AA.AC;
VAR AA.VSAB. AA.VSAC. AB.VSAC.;
%MEND PHWT;

DATA TEST; /* EXAMPLE OF IMPLEMENTATION AND RESULTS */
INPUT BLOCK A B C;
CARDS;
1 894 818 387
2 844 800 980
3 738 729 974
4 621 551 621
5 492 472 598
6 248 206 295
7 167 112 214
8 161 98 179
9 92 71 103
;
%MEND FRIEDMAN;

%RANKSUM (TEST, A B, C);
%FRIEDMN (A, B, C, TRT = 3, BLK = 9, ALPHA = 0.05);
### APPENDIX II

**FRIEDMAN TEST**

**NONPARAMETRIC RANDOMIZED BLOCK DESIGN**

<table>
<thead>
<tr>
<th>ASUM</th>
<th>BSUM</th>
<th>CSUM</th>
<th>TOTSUMSQ</th>
<th>SUMT3SUM</th>
<th>SUM2SUM</th>
<th>SUM1I</th>
<th>K</th>
<th>B</th>
<th>ALPHA</th>
</tr>
</thead>
<tbody>
<tr>
<td>19.5</td>
<td>9</td>
<td>25.5</td>
<td>1111.5</td>
<td>8</td>
<td>2</td>
<td>6</td>
<td>3</td>
<td>9</td>
<td>0.05</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.972222</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>FR</th>
<th>P _FR</th>
<th>FRADJ</th>
<th>P _FRADJ</th>
</tr>
</thead>
<tbody>
<tr>
<td>15.5</td>
<td>0.000430743</td>
<td>15.9429</td>
<td>0.000345186</td>
</tr>
</tbody>
</table>

**NONPARAMETRIC MULTIPLE COMPARISON PROCEDURE**

**LEAST SIGNIFICANT DIFFERENCES**

<table>
<thead>
<tr>
<th>K</th>
<th>B</th>
<th>ALPHA</th>
<th>Z</th>
<th>RHS</th>
<th>ASUM</th>
<th>BSUM</th>
<th>CSUM</th>
<th>AB</th>
<th>AC</th>
<th>BC</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>9</td>
<td>0.05</td>
<td>2.39398</td>
<td>10.1568</td>
<td>19.5</td>
<td>9</td>
<td>25.5</td>
<td>10.5</td>
<td>6</td>
<td>16.5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>AVSB</th>
<th>AVSC</th>
<th>BVSC</th>
</tr>
</thead>
<tbody>
<tr>
<td>SIGNIFICANCE</td>
<td>NO SIGNIFICANCE</td>
<td>SIGNIFICANCE</td>
</tr>
</tbody>
</table>